General limits for entanglement distribution

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Outline

1. Introduction
    - Quantum entanglement
    - Quantum discord
2. Results
    - General protocol for entanglement distribution
    - Limits for entanglement distribution imposed by discord
    - Limits for entanglement distribution with separable states

Outline

Entanglement

- Entangled states are states which *cannot* be written as[1]

$$\rho^{AB} = \sum_i p_i \, |a_i\rangle\langle a_i|^A \otimes |b_i\rangle\langle b_i|^B \,. \tag{1}$$

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- Entangled states *cannot* be prepared by local operations and classical communication (LOCC).
- Entanglement plays an important role for several tasks in quantum information theory: quantum cryptography[2], quantum dense coding[3], and quantum teleportation[4].

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Quantifying entanglement

- Relative entropy of entanglement[5]:

$$E_R^{A|B}\left(\rho^{AB}\right) = \min_{\sigma^{AB}\in\mathcal{S}} S\left(\rho^{AB}\|\sigma^{AB}\right), \tag{2}$$

with $S(\rho\|\sigma) = \mathrm{Tr}[\rho\log\rho] - \mathrm{Tr}[\rho\log\sigma]$. $E_R^{A|B}$ is an upper bound on the distillable entanglement and a lower bound on the entanglement of formation.

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- Logarithmic negativity[6]:

$$E_N^{A|B}(\rho^{AB}) = \log_2 \|\rho^{T_A}\|_1 \tag{3}$$

  with partial transposition T_A and trace norm $\|M\|_1 = \mathrm{Tr}\sqrt{M^\dagger M}$. E_N is an upper bound on the distillable entanglement and a lower bound on the PPT entanglement cost.

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Quantum discord

- A state has nonzero quantum discord if it *cannot* be written as[7]

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with $\langle i|j\rangle = \delta_{ij}$.

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- Compare to separable states (i.e. states without entanglement):

  $$\rho^{AB} = \sum_i p_i \left|a_i\right\rangle\left\langle a_i\right|^A \otimes \left|b_i\right\rangle\left\langle b_i\right|^B . \tag{5}$$

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- Properties:
  - Separable states can have nonzero quantum discord.
  - Quantum discord is *not* symmetric under permutations.

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Quantifying discord

- The original "quantum discord"[1]:

$$\delta^{A|B}\left(\rho^{AB}\right) = S\left(\rho^{A}\right) - S\left(\rho^{AB}\right) + \min_{\{\Pi_i^A\}} \sum_i p_i S\left(\rho_i\right), \qquad (6)$$

the minimum is taken over all von Neumann measurements Π_i^A on A with $p_i = \mathrm{Tr}\left[\Pi_i^A \rho^{AB} \Pi_i^A\right]$, and $\rho_i = \Pi_i^A \rho^{AB} \Pi_i^A / p_i$.
Interpretation: difference of two mutual informations.

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  Interpretation: difference of two mutual informations.

- Relative entropy of discord[2] (called "discord" in the following):

$$\Delta^{A|B}\left(\rho^{AB}\right) = \min_{\{\Pi_i^A\}} S\left(\rho^{AB} \| \sum_i \Pi_i^A \rho^{AB} \Pi_i^A\right) \qquad (7)$$

  with the relative entropy $S\left(\rho \| \sigma\right) = \mathrm{Tr}\left[\rho \log \rho\right] - \mathrm{Tr}\left[\rho \log \sigma\right]$.
  Interpretation: amount of information which cannot be localized via classical communication from Alice to Bob.

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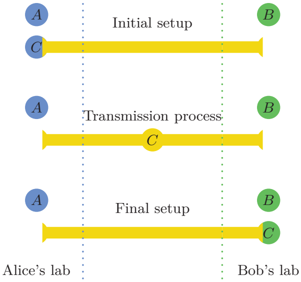
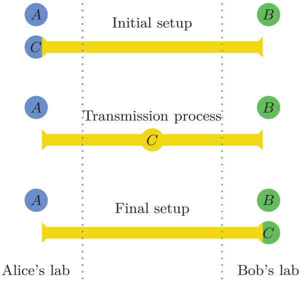
Outline

General protocol for entanglement distribution



Alice and Bob share a mixed state $\rho = \rho^{ABC}$.

General protocol for entanglement distribution



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- Initially Alice has AC, and Bob has B, the initial entanglement is $E_{\mathrm{initial}} = E^{AC|B}(\rho)$.

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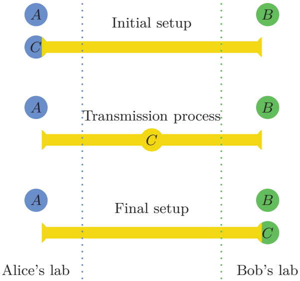

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- Alice sends the particle C to Bob via a perfect quantum channel.

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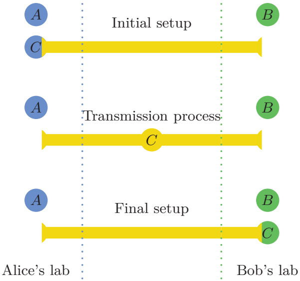

Alice and Bob share a mixed state $\rho = \rho^{ABC}$.

- Initially Alice has AC, and Bob has B, the initial entanglement is $E_{\mathrm{initial}} = E^{AC|B}(\rho)$.
- Alice sends the particle C to Bob via a perfect quantum channel.
- The final entanglement is $E_{\mathrm{final}} = E^{A|BC}(\rho)$.

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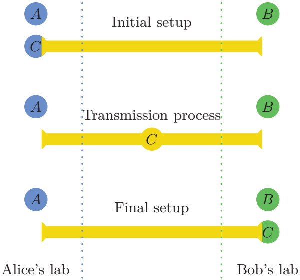
Limits for entanglement distribution imposed by discord



Theorem

Quantum discord between the exchanged particle C and the rest of the system AB limits the amount of distributed entanglement[3]*:*

$$E^{A|BC} - E^{AC|B} \leq \Delta^{C|AB}. \tag{8}$$

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[3] A. S., H. Kampermann, and D. Bruß, PRL **108**, 250501 (2012); T. K. Chuan *et al.*, PRL **109**, 070501 (2012).

Limits for entanglement distribution imposed by discord

Theorem

Quantum discord between the exchanged particle C and the rest of the system AB limits the amount of distributed entanglement:

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The result holds for any distance-based quantifier of entanglement and discord:

$$E^{X|Y}(\rho^{XY}) = \min_{\sigma^{XY} \in \mathcal{S}} D(\rho^{XY}, \sigma^{XY}), \tag{10}$$

$$\Delta^{X|Y}(\rho^{XY}) = \min_{\{\Pi_i^X\}} D(\rho^{XY}, \sum_i \Pi_i^X \rho^{XY} \Pi_i^X). \tag{11}$$

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D can be the relative entropy $S(\rho||\sigma) = \mathrm{Tr}[\rho \log \rho] - \mathrm{Tr}[\rho \log \sigma]$, or any distance which satisfies the triangle inequality and does not increase under quantum operations.

Applications of the theorem

- Relation between the amount of entanglement in different bipartitions of the system:

$$\left|E^{A|BC} - E^{AC|B}\right| \leq \Delta^{C|AB}. \tag{12}$$

$\Rightarrow$ for a small amount of discord $\Delta^{C|AB} = \varepsilon$ the amount of entanglement $E^{A|BC}$ differs from $E^{AC|B}$ at most by ε.

Applications of the theorem

- Relation between the amount of entanglement in different bipartitions of the system:

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$\Rightarrow$ for a small amount of discord $\Delta^{C|AB}=\varepsilon$ the amount of entanglement $E^{A|BC}$ differs from $E^{AC|B}$ at most by ε.

- Relation between distillable entanglement and entanglement cost:

$$E_d^{A|BC}-E_c^{AC|B}\leq\Delta^{C|AB}. \tag{13}$$

$E_d^{A|BC}-E_c^{AC|B}$ corresponds to the number of singlets gained in the process of entanglement distribution in the asymptotic limit.

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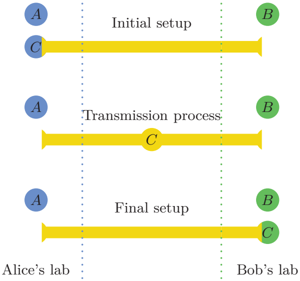

For successful entanglement distribution the exchanged particle *C* does not have to be entangled with the rest of the system *AB*:

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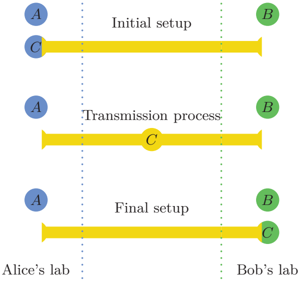
Limits for entanglement distribution with separable states



For successful entanglement distribution the exchanged particle *C* does not have to be entangled with the rest of the system *AB*:

- There exist states $\rho = \rho^{ABC}$ such that $E^{C|AB}(\rho) = 0$ and $E^{A|BC}(\rho) > E^{AC|B}(\rho)$.

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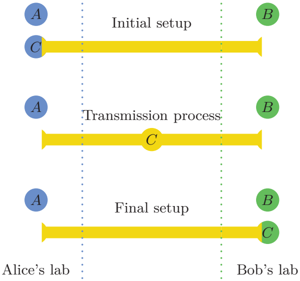

For successful entanglement distribution the exchanged particle C does not have to be entangled with the rest of the system AB:

- There exist states $\rho = \rho^{ABC}$ such that $E^{C|AB}(\rho) = 0$ and $E^{A|BC}(\rho) > E^{AC|B}(\rho)$.
- The process is then called "entanglement distribution with separable states".[4]

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Limits for entanglement distribution with separable states

- In the following we quantify the entanglement by the logarithmic negativity $E_N^{X|Y}(\rho^{XY}) = \log_2 \|\rho^{T_X}\|_1$.

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Limits for entanglement distribution with separable states

- In the following we quantify the entanglement by the logarithmic negativity $E_N^{X|Y}(\rho^{XY}) = \log_2 ||\rho^{T_X}||_1$.

- Any rank two state $\rho = \rho^{ABC}$ which is separable between *AB* and *C* satisfies the following equality:

$$E_N^{A|BC}(\rho) = E_N^{AC|B}(\rho). \tag{14}$$

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Theorem

Entanglement distribution with separable states requires states with rank at least three.[5]

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Limits for entanglement distribution with separable states

Further results[6]

- Separable states of the form

$$\rho^{ABC} = p \cdot \rho_1^{AB} \otimes \rho_1^C + (1-p) \cdot \rho_2^{AB} \otimes \rho_2^C \tag{15}$$

  can only be used for entanglement distribution if the transmitted particle C has at least dimension three, and if both states ρ_1^C and ρ_2^C are not pure.

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- If the transmitted particle C is a qubit, a general separable state

$$\rho^{ABC} = \sum_i p_i \cdot \rho_i^{AB} \otimes \rho_i^C \tag{16}$$

  can only be used for entanglement distribution if the Bloch vectors of ρ_i^C are not all in the same plane.

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Summary

- Entanglement distribution is in general limited by quantum discord: $E^{A|BC} - E^{AC|B} \leq \Delta^{C|AB}$.

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- Stronger limits were presented for entanglement distribution with separable states:
  - Entanglement distribution with separable states requires states with rank at least three.
  - Mixtures of two product states can only be used for entanglement distribution if the exchanged particle has at least dimension three.

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- Classical analogy to entanglement distribution with separable states: secrecy can be distributed by sending nonsecret correlations through a private channel.[7]

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- Classical analogy to entanglement distribution with separable states: secrecy can be distributed by sending nonsecret correlations through a private channel.[7]
- Experimental demonstration for entanglement distribution with separable states was also presented recently.[8][9]

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