Universal computation by multi-particle quantum walk in 2D

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Introduction

Quantum computation consists of L gates $U_1, ..., U_L$ drawn from the universal gate basis. The action of L-gate quantum circuit on an n-qubit pure state $|\psi\rangle$ is

$$U_L ... U_1 |\psi\rangle .$$

Feynman [**?**] introduced *"clock"-spins* each of which passes next to each gate in turn. c_k is the lowering operator of the k-th clock qubit. Feynman Hamiltonian that acts on both the input state $|\psi\rangle$ and $L+1$ clock qubits

$$H_F = F + F^\dagger = \sum_{t=1}^{L} U_t \otimes c_t^\dagger c_{t-1} + U_t^\dagger \otimes c_{t-1}^\dagger c_t .$$

If $|t\rangle = |0...010...0\rangle$ denotes the *pulse clock*, where '1' is at t-th position, we can write the Hamiltonian as

$$H_F = \sum_{i=1}^{L} U_t \otimes |t\rangle \langle t-1| + U_t^\dagger \otimes |t-1\rangle \langle t| .$$

Fixing the initial state $|\psi_0\rangle$ of the data qubits, define a basis

$$|\phi_t\rangle = |\psi_t\rangle \otimes |t\rangle = U_t ... U_1 |\psi_0\rangle \otimes |t\rangle .$$

Then Feynman Hamiltonian has a simple form of a Hamiltonian of a quantum walk on the "line" of states $|\phi_t\rangle$

$$H_F = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 \\ & & & & & 1 & 0 \end{bmatrix}$$

After some time measure the clock register. When we obtain 'L' the state of the system after the measurement becomes $|\phi_L\rangle = |\psi_L\rangle \otimes |L\rangle$. The data register contains the desired output of the circuit U.
Feynman proved that evolution for time L under H_F with the clock initialized in $|10...0\rangle$ will efficiently approximate a sequenced implementation of circuit.

Modify Feynman's Hamiltonian

$$H_{cir} = \sum_{t=1}^{L} H_t = \sum_{t=1}^{L} -U_t \otimes |t\rangle\langle t-1| - U_t^\dagger \otimes |t-1\rangle\langle t| + |t\rangle\langle t| + |t-1\rangle\langle t-1|,$$

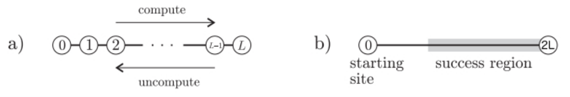
so that the history state, which encodes the progress of the quantum circuit U,

$$|\phi_{history}\rangle = \frac{1}{\sqrt{L+1}} \sum_{t=0}^{L} U_t..U_1 |\psi_0\rangle \otimes |t\rangle.$$

is a ground state of H_{cir}.
To increase probability of getting the desired output state, one can add L identity gates at the end of the circuit U, following **Nagaj** [**?**],

$$U' = \mathbb{1}_{2L}...\mathbb{1}_{L+1}U,$$

and, accordingly, expand the clock register to hold states $|0\rangle, ..., |2L\rangle$. All the states $|\psi_t\rangle$ of the data register with $t \geq L$ are now tha same and equal to the desired output state $|\psi_L\rangle$.
Let the system evolve for a time τ chosen uniformly at random between 0 and $O(L\log^2 L)$. The probability of measuring the clock register with $t \in [L, 2L]$ at any time τ is $1/2$.



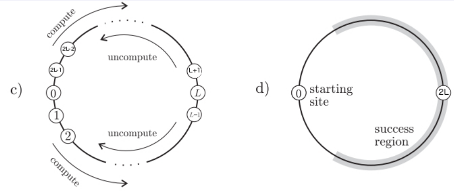
Consider a circular clock register. For the gates U_t with $t = L+1, ..., 2L$ we choose $U_t = U^\dagger_{2L+1-t}$, which leaves the history state

$$|\phi_{history}\rangle = \frac{1}{\sqrt{2L}} \sum_{t=0}^{2L-1} U_t..U_1 |\psi_0\rangle \otimes |t\rangle$$

a ground state of the Hamiltonian

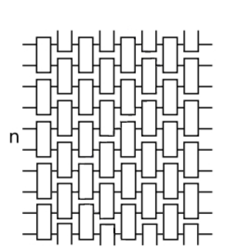
$$H_{cir} = \sum_{t=1}^{2L-1} H_t = \sum_{t=1}^{2L-1} -U_t \otimes |t\rangle\langle t-1| - U_t^\dagger \otimes |t-1\rangle\langle t| + |t\rangle\langle t| + |t-1\rangle\langle t-1| .$$

To increase the probability of completing the circuit after measuring the clock register after some time, we, as before, add identity gates at the end of the circuit U in the middle of the circular region.



Model under investigation

Introduced by B. Terhal and N. Breuckmann [**?**]. The circuit is considered on n qubits with one- and two-qubit gates U_i, $i = 1, ..., S$. The depth of the circuit is $D \leq S$.



For each qubit q in the original circuit, we define a clock register $|t\rangle_q$ with $t = 0, ..., D$.

For every single-qubit gate $U_t^1[q]$ acting on qubit q at time-step t, there is a term $H_t^1[q]$ in H_{cir}

$$H_t^1[q] = -(U_t^1[q] \otimes |t\rangle\langle t-1|_q + h.c.) + |t\rangle\langle t|_q + |t-1\rangle\langle t-1|_q \,.$$

For every two-qubit gate $U_t^2[q,p]$ acting on qubits q and p at time t, there is a term

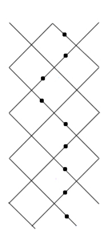
$$H_t^2[q,p] = -(U_t^2[q,p] \otimes |t,t\rangle\langle t-1,t-1|_{q,p} + h.c.) + |t,t\rangle\langle t,t|_{q,p} + |t-1,t-1\rangle\langle t-1,t-1|_{q,p}$$

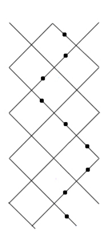
The circuit Hamiltonian

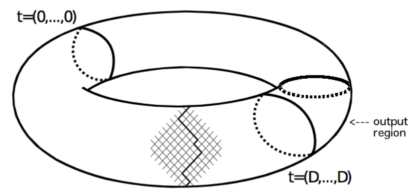
$$H_{cir} = \sum_{t=1}^{D} H_t,$$

is then the sum of H_t, which is a sum of one- and two-qubit gates done in parallel at time t.

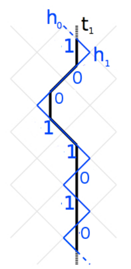
Put n qubits forming a chain on a lattice in the middle of each edge. Every time two qubits have the same time, i.e. on the same vertical line, a gate is done and both qubits jump forward (or backward).







The lattice covers a torus of hight n and circumference $2D$. The output region is the strip of the torus with time $t \geq D$ where the corresponding identity gates are.



Relabel a valid time configuration $|t\rangle = |t_1, ..., t_n\rangle$ as a pair (τ, x), where $\tau \in \mathbb{Z}_D$ and $x = (x_1, .., x_n)$ is a bit string.

Set $\tau = \left[\frac{t_1}{2}\right]$ and x is a "deviation" string.

The circuit Hamiltonian is unitary equivalent to

$$H_{cir} = \sum_{i=1}^{n} (\sigma_i^+ \sigma_i^- \sigma_{i+1}^- \sigma_{i+1}^+ + h.c.) + (\sigma_n^+ \sigma_n^- \sigma_1^- \sigma_1^+ + h.c.)$$
$$- \sum_{i=1}^{n} (\sigma_i^- \sigma_{i+1}^+ + h.c.) - (\sum_{\tau=0}^{D-1} \sigma_1^- \sigma_n^+ \otimes |\tau - 1\rangle \langle \tau| + h.c.)$$

Terhal and Breuckmann [**?**] proved that the gap of H_{cir} is appropriately lower-bounded.

Theorem

The smallest non-zero eigenvalue λ_1 of H_{cir} with $D > \frac{n}{2}$, is bounded

$$\lambda_1(H_{cir}) \geq \frac{\pi^4}{4D^2(n-1)n} + O(\frac{1}{n^4 D^2}).$$

And constructed a map from any class of problems $L = L_{yes} \cup L_{no}$ in QMA to a Hamiltonian such that:

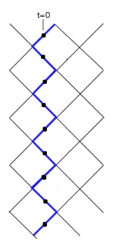
- if $x \in L_{yes}$, then the Hamiltonian $H(x)$ has eigenvalue lower than or equal to some a.
- if $x \in L_{no}$, then all eigenvalues of the Hamiltonian are larger than or equal to b, where $|a - b| \geq \frac{1}{\text{poly}(n)}$.

proving that 'local Hamiltonian' problem for this model is QMA-complete.

The quantum walk of the chain of atoms on a torus is governed by the Hamiltonian

$$H_{qw} = -\sum_{i=1}^{n-1} \sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ - \sum_{\tau=0}^{D-1} \sigma_1^- \sigma_n^+ \otimes |\tau - 1\rangle \langle \tau| + \sigma_1^+ \sigma_n^- \otimes |\tau\rangle \langle \tau - 1|$$

with the condition $\sum_{i=1}^{n} Z_i = 0$.



The initial state is when all times set to 0, or equivalently, $\tau = 0$ and $x_0 = (01010....)$.

Two questions:

- How long do we have to wait to finish computation for one qubit, i.e. to find τ in an output region, $I_{out} = [\Gamma_1, \Gamma_2] \subset [0, D]$ with high probability.
- How long do we have to wait to finish computation for all qubits, i.e. to find the <u>the whole</u> string in the vertical strip on a cylinder corresponding to time $[\Gamma_1, \Gamma_2] \subset [0, D]$.

Average distribution

$$p_{I_{out}}(T) = \frac{1}{T} \int_0^T \mathrm{Tr}\, P_{I_{out}} e^{-iH_{qw}t} |\tau = 0, x_0\rangle \langle \tau = 0, x_0| e^{iH_{qw}t} dt.$$

Mixing time

$$M_t = \min\{T \,|\, \forall t \geq T, |p_{I_{out}}(T) - p_{I_{out}}(\infty)| \leq \epsilon\}.$$

The eigenstates with respect to the counter variable τ are

$$|\psi_k\rangle = \frac{1}{\sqrt{D}} \sum_{\tau=0}^{D-1} e^{2\pi i k\tau/D} |\tau\rangle\,,\ k = 0, ..., D-1,$$
$$H_{qw}\,|\mu\rangle \otimes |\psi_k\rangle = (H(k)\,|\mu\rangle) \otimes |\psi_k\rangle\,,$$

where

$$H(k) = -\sigma_1^- \sigma_n^+ e^{2\pi i k/D} - \sigma_1^+ \sigma_n^- e^{-2\pi i k/D} - \sum_{i=1}^{n-1} (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+).$$

Making Jordan-Wigner and Fourier transformation,

$$H(k) = \sum_{m=1}^{n} e_m(k) b_m^*(k) b_m(k),$$

where $e_m(k) = -2\cos\frac{2\pi}{n}(m - \frac{k}{D})$ and $b_m^*(k)$ is a fermionic creation operator. Therefore,

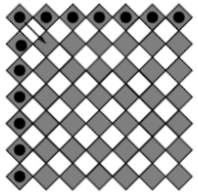
$$H_{cir} = \sum_{k,\alpha} E_\alpha(k)\,|\mu_\alpha(k)\rangle\,\langle\mu_\alpha(k)| \otimes |\psi_k\rangle\,\langle\psi_k|$$

where the sum is over all $0 \le k \le D-1$ and all subsets $\alpha \subset \{1, ..., n\}$ of size $n/2$ and $|\mu_\alpha(k)\rangle = b_{\alpha_1}^*(k) ... b_{\alpha_{n/2}}^*(k)\,|0...0\rangle$. Here

$$E_\alpha(k) = \sum_{j=1}^{n/2} e_{\alpha_j}(k).$$

Janzing's model [?]

On a $n \times n$ lattice consider a chain of atoms.



The interactions between sites (i, j) and $(i+1, j+1)$ which generate independent hopping of atoms

$$K = \sum_{i,j=1}^{n-1} a_{i,j} a^{\dagger}_{i+1,j+1} + h.c.$$

To make the chain connected introduce a strong attractive force between neighboring sites

$$H_{pot} = -E \sum_{<(i,j),(k,l)>} N_{i,j} N_{k,l} + E_0.$$

Here $N_{i,j} = a^{\dagger}_{i,j} a_{i,j}$ and the sum runs over all neighboring sites, i.e. $|i-k| + |j-l| = 1$. Then the synchronization Hamiltonian is

$$H = K + H_{pot}.$$

A Hilbert space $\mathcal{H}_c$ is spanned by all allowed configurations of unbroken chain. The error of the replacement H with H_{eff} is small.

$$H_{eff} = \sum_{i,j=1}^{n-1} a_{i,j} a^{\dagger}_{i+1,j+1} N_{i,j+1} N_{i+1,j} + h.c. \Big|_{\mathcal{H}_c}.$$

H_{eff} can be represented as

$$\begin{matrix} a & \otimes & N \\ \otimes & & \otimes \\ N & \otimes & a^{\dagger} \end{matrix} + h.c.$$

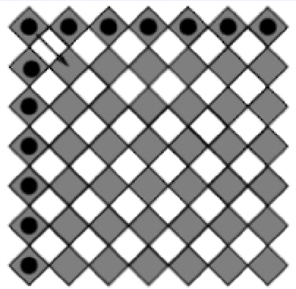
Holonomic quantum computing. A Hamiltonian of a type

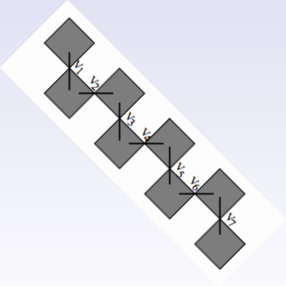
$$G(t) = e^{iXt} G_0 e^{iXt},$$

with $t \in [0, T]$, where X is a self-adjoint operator is changed adiabatically on a closed loop, i.e. $G(0) = G(T)$, such that the overall effect is a unitary which depends only on the loop and not on the speed of the change of $G(t)$.
At each lattice site replace a qubit by a qutrit in either one of three states $|0\rangle, |\uparrow\rangle, |\downarrow\rangle$.
Two adjacent spin particles encode one qubit with logical states $|0\rangle$, $|1\rangle$ by

$$|0\rangle = |\downarrow\rangle \otimes |\uparrow\rangle \text{ and } |1\rangle = |\uparrow\rangle \otimes |\downarrow\rangle .$$

One-qubit gates. The whole region which carries the gate consists of a stripe of width 2.





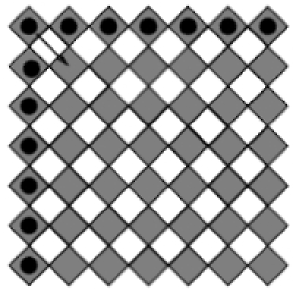
Inside the strip the interactions are chosen as follows

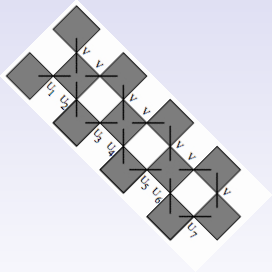
$$V_j = e^{2\pi\, i \frac{j-1}{l-1} X} (\sigma_z \otimes N + N \otimes \sigma_z) e^{-2\pi\, i \frac{j-1}{l-1} X},$$

here N is the number operator and $X = \sigma_x (or\ \sigma_y) \otimes (\sigma_x \cos\phi + \sigma_z \sin\phi)$ with arbitrary angle ϕ.

After passing the interaction region, a gate $\exp\{2\pi\, i \cos\phi\, \sigma_x\}$ (or $\exp\{2\pi\, i \cos\phi\, \sigma_y\}$) is done.

Two-qubit gates. Consider interaction strips that consist of 3 adjacent rows.





An interaction in the strip is $V \otimes \mathbb{1} + \mathbb{1} \otimes U_j$, with $j = 1, ..., l$.
Chose $V = \sigma_z \otimes N$ and

$$U_j = \sigma_z \otimes |\downarrow\rangle \langle\downarrow| + (e^{2\pi\, i \frac{j-1}{l-1} \tilde{X}} \sigma_z e^{-2\pi\, i \frac{j-1}{l-1} \tilde{X}}) \otimes |\uparrow\rangle \langle\uparrow|,$$

where $\tilde{X} = \sigma_x \cos\phi + \sigma_z \sin\phi$ with arbitrary angle ϕ.

After passing the interaction region, a gate controlled-exp$\{i \sin\phi\, \sigma_z\}$ is done.

Example

C-NOT gate can be done by doing two controlled-phase gate with $\sin\phi_z = \frac{\pi}{4}$ and one σ_y rotation with $\cos\phi_y = \frac{1}{4}$. Since

$$NOT = e^{i\frac{3}{2}\pi} R_z(\pi) R_y(\pi),$$

we get

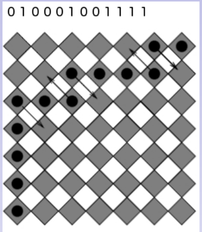
$$\text{C-NOT} = e^{i\frac{3}{2}\pi} \circ \text{c-}e^{i\frac{\pi}{4}\sigma_z} \circ \text{c-}e^{i\frac{\pi}{4}\sigma_z} \circ e^{2\pi i \frac{1}{4}\sigma_y} \otimes \mathbb{1}.$$

The complete Hamiltonian of the system is

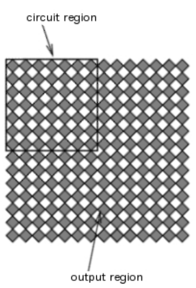
$$\hat{H} = H_{pot} + K + \sum_{<(i,j),(i',j')>} W_{(i,j),(i',j')}.$$

(Here H_{pot} and K are adjusted for qutrits:
$K = \sum_{i,j=1}^{n-1} a_{i,j,\downarrow} a^\dagger_{i+1,j+1,\downarrow} + a_{i,j,\uparrow} a^\dagger_{i+1,j+1,\uparrow} + h.c.$ and $N = a^\dagger_{i,j,\downarrow} a_{i,j,\downarrow} + a^\dagger_{i,j,\uparrow} a_{i,j,\uparrow}$).
Here $W_{(i,j),(i',j')}$ are only non-zero if the site (i, j) is adjacent to site (i', j') and if the pair belongs to the same interaction stripe. Inside the strips they are given by V_j, V, U_j as before.

The positions of the atoms define a binary word of length $2n - 2 = 2m$. A symbol 0 at jth means that jth and $j+1$-th atoms form a horizontal line. A symbol 1 at jth means that jth and $j+1$-th atoms form a vertical line.



A *circuit region* is a square of length k that confines all interaction strips. The complement of the circuit region is the *output region* since we will read out the result of the computation there.



The time needed by the atoms to pass the circuit region for the first time is linear in k.

Theorem

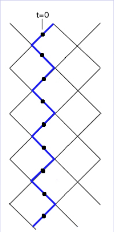
There is a time $t \in O(k)$ such that the probability of finding all atoms outside the circuit region is at least $1 - 12/k$ given that the size of the whole lattice is sufficiently large.

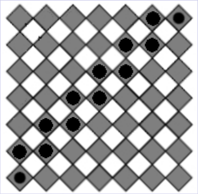
At random time instant the probability of finding all atoms in the output region is high.

Theorem

Let the circuit region be a square of length $k = (n-1)/4$. Then the probability of finding all atoms outside the circuit region tends to 1 for $n \to \infty$.

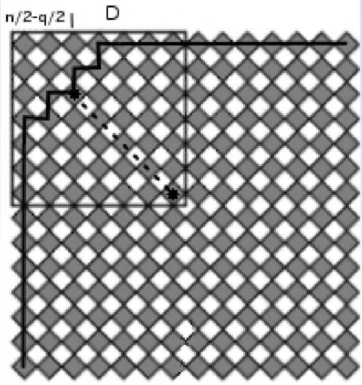
Correspondence between quantum walks





The vertices on our lattice correspond to the lattice sites of the Janzing's model.

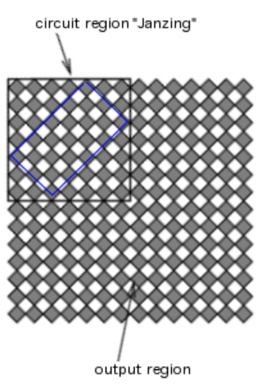
The initial state $\tau = 0$ and $x = (01010....)$, correspond to the following initial state in Janzing's model.



Finishing computation for one qubit q, in Janzing's model it corresponds to moving this qubit D positions down.

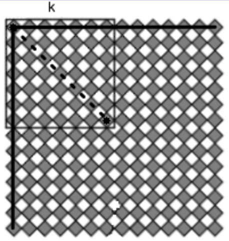
Using Janzing's analysis it can be shown that it takes $t \in O(n/4 + D)$ to achieve a hight probability of finding the $q = n/2$-th qubit outside the circuit region for the first time.

To finish computation for all qubits, in Janzing's model it corresponds to finding all atoms outside the blue region.



Following the same analysis we find that for $k = n/2 + D$ it takes $t \in O(k)$ time to wait to have a high probability of finding all atoms outside the circuit region.

Proof



The initial state is

$$|I\rangle = |0...01...1\rangle \,,$$

where there are m symbols "0" and "1"
The middle atom moved k positions down iff there are at least k symbols "1" in the first m positions of the chain.

Define the operator

$$N = \sum_{j=1}^{m} b_j^\dagger b_j.$$

Chebyshev inequality

$$P\Big(|N - E_t(N)| \geq \epsilon\Big) \leq \frac{V_t(N)}{\epsilon^2}.$$

The expectation value of N after time t is

$$E_t(N) = \sum_{j=1}^{m} \langle I_t|\, b_j^\dagger b_j \,|I_t\rangle \,.$$

Each term $\langle I_t|\, b_j^\dagger b_j \,|I_t\rangle = \sum_{l=1}^{2m} u_{jl;t}\overline{u_{jl;t}}\, \langle l|\, b_j^\dagger b_j \,|l\rangle = \sum_{l=m+1}^{2m} |u_{jl;t}|^2$ is a sum of the probabilities $|u_{jl;t}|^2$ for a particle starting at site l to be found at site j after time t in a singular particle quantum walk on a line.

For each

$$m \leq l \leq m + 4k$$

we have to wait only the time $O(k)$ in order to achieve the width of the wave function of a particle starting at position l larger than $4k$. The probability of finding it on the left half is larger than $1/3$.

There are $4k$ l's that satisfy the above restriction. Therefore, choose a time $t \in O(k)$ such that

$$E_t(N) = 4k/3.$$

It can be shown that $V_t(N) \leq E_t(N)$, so choosing $\epsilon = k/3$ we get

$$P\Big(|N - \frac{4}{3}k| \geq \frac{k}{3}\Big) \leq \frac{V_t(N)}{\epsilon^2} \leq \frac{4k}{3\epsilon^2} = \frac{12}{k}.$$

Q.E.D.

Conclusion

- **Motivation:** To come up with quantum mechanically plausible model suitable for universal quantum computation.
- **Model:** Quantum walk on a torus.
- **Improvements:** One atoms encodes one qubit; no need for multiple steps (the whole strip) to do one simple gate.
- **Disadvantage:** Complicated spectrum; torus structure; use of clock qubits.
- **Open questions:** Adiabatic computation.

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Thank you!