Difficult instances of the counting problem for 2-quantum-SAT are very atypical

Niel de Beaudrap

CWI, Amsterdam

DIQIP/QALGO joint meeting
14 May 2014
preamble: about frustration-freeness

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- more difficult to study the ground states

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DIQIP/QALGO joint meeting 2 / 23
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\[ H = \sum_{a,b} \sigma_a^{(z)} \otimes \sigma_b^{(z)} \]

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\[ |\uparrow\downarrow\rangle \text{ and } |\downarrow\uparrow\rangle \]
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preamble: frustration-freeness and satisfiability

Consider two-body interactions on spin-1/2 systems (qubits).

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In the special case where

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for \( x, y \in \{0, 1\} \), and similarly for all terms:

Avoiding local “classical” two-bit configurations is exactly

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Determining whether a two-body Hamiltonian on qubits is

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(a quantum generalization which one may study for its own sake)

How difficult are these problems?

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(Complexity of \( 3\text{-SAT} / 3\text{-QSAT} \) (3-body constraints): . . . very hard.)

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- Generalizes #2-SAT: counting solutions to an instance of 2-SAT
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How difficult is it to solve #2-SAT/#2-QSAT?

- Complexity of #2-SAT: #P-complete
  (hard as counting solutions to NP-complete problems)

Entangled avoided configurations?

Product states form a set of measure zero

"entangled constraints" act similarly to a pair of product constraints

More fundamentally: constraints in multiple local bases

Monotonicity seems to be the source of difficulty of #2-SAT.

The sharp decline in difficulty of #2-QSAT provides evidence for this: monotonicity requires a preferred basis.
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summary of results

**Intuition.** “The only families of random \( \#2\text{-QSAT} \) which are ‘hard’ are those which, asymptotically, are broadly similar to hard families of \( \#2\text{-SAT} \).”
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Looking for the boundaries of the hard instances:
- Only allow product constraints:
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(b) Erdős–Rényi graphs.
Then as $q \rightarrow 0$, ‘hard’ instances of #2-QSAT become unlikely for any fixed density of interaction terms.
Outline

1. Constructing models of random #2-QSAT
   - Erdős–Rényi graphs and percolated lattices
   - Random instances of 2-QSAT on random graphs
   - Random frustration-free Hamiltonians on random graphs
   - Common features of the interaction graph models

2. Analysis of #2-QSAT on random graphs
   - Effective long-range constraints
   - Onset of frustration in random two-body Hamiltonians on qubits
   - Frozen subsystems in frustration-free models

3. Summary
models of random graphs (part 1)

Prior work on random 2-satisfiability:

- random 2-SAT with uniformly random boolean constraints  
  *e.g.*, [Chvátal+Reed, 1992]

- random 2-QSAT with Haar-uniform constraints  
  *e.g.*, [Laumann et al., 2010]

Both of these consider randomly constructed examples of 2-SAT/2-QSAT on Erdős–Rényi graphs: graph with \(n\) labelled vertices (\(\sim\) boolean variables/spins) \(m\) edges out of a possible \(\binom{n}{2}\) are selected for inclusion—essentially the same as including edges i.i.d. with probability \(p = \frac{m}{\binom{n}{2}}\) a two-site constraint is associated to each edge.

These graphs have “as little structure as possible”—while perhaps unphysical, this (and prior work) motivates this model.

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We consider edge-percolated square/cubic lattice models for 2-QSAT:
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Results for square/cubic lattices will have analogues for other lattices.
random two-body hamiltonians on qubits

How to build a random two-body spin-1/2 Hamiltonian, on $n$ qubits.

1. Choose a random graph model, and a density $\gamma = m/n$ for the interaction graph.
2. Choose a distribution for the constraints.

$\rightarrow$ distribution of $n$-qubit Hamiltonians depending on $\gamma$ and $q$. 

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   - Independent factor product constraints:
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1. Choose a random graph model, and a density $\gamma = m/n$ for the interaction graph.

2. Choose a distribution for the constraints.
   - Monotone 2-SAT: point-mass function on $|0\rangle\langle 0| \otimes |0\rangle\langle 0|$
   - “Random 2-SAT” — [Chvátal+Reed, 1992]: projector $|a\rangle\langle a| \otimes |a'\rangle\langle a'|$ for $a, a' \in \{0, 1\}$ uniformly i.i.d.
   - Independent factor product constraints: projector $|\alpha\rangle\langle \alpha| \otimes |\alpha'\rangle\langle \alpha'|$ for $|\alpha\rangle, |\alpha'\rangle \in \{|\alpha_1\rangle, |\alpha_2\rangle, \ldots\}$ i.i.d. according to $q = (q_1, q_2, \ldots)$, where $q_k = \Pr[\alpha = \alpha_k]$

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$\rightarrow$ distribution of $n$-qubit Hamiltonians depending on $\gamma$ and $q$. 
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If we over-constrain the spin-system, it may change abruptly from being frustration-free to frustrated.

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how the interaction graphs behave

The complexity of solving \#2-QSAT on these Hamiltonians will be largely (but not completely) governed by the interaction graphs. The following results hold \textit{almost surely as } $n \to \infty$ (that is, probability $1 - o(1/n)$ for a randomly constructed graph):
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  \( \gamma_0 = \frac{1}{2} \) Erdős–Rényi graphs, \( \gamma_0 = 1 \) square lattices, \( \gamma_0 \approx 0.7464 \) cubic lattices

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Outline

1. Constructing models of random \#2-QSAT
   - Erdős–Rényi graphs and percolated lattices
   - Random instances of 2-QSAT on random graphs
   - Random frustration-free Hamiltonians on random graphs
   - Common features of the interaction graph models

2. Analysis of \#2-QSAT on random graphs
   - Effective long-range constraints
   - Onset of frustration in random two-body Hamiltonians on qubits
   - Frozen subsystems in frustration-free models

3. Summary
effective next-nearest-neighbor constraints

- We describe constraints $h_{ab} = |\eta_{ab}\rangle\langle\eta_{ab}|$
in terms of the state $|\eta_{ab}\rangle$ in its support.
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  induce another constraint $|\tilde{\eta}_{ac}\rangle$ on $\{a, c\}$:

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- For constraints $|\eta\rangle = |\alpha\rangle |\alpha'\rangle$ whose factors are distributed \textit{i.i.d.}
  according to $q$, $|\tilde{\eta}_{ac}\rangle \neq 0$ happens with probability

  $$Q := \sum_{k \geq 1} q_k (1 - q_k) = 1 - \sum_{k \geq 1} q_k^2 = 1 - \|q\|_2^2.$$
long-range constraints and effective graph density

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N.B. \( Q \to 1 \) as the constraints become less monotone \( (q \to 0) \).
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**Ansatz:** long range constraints in a Hamiltonian with density \( \gamma > 0 \) act like connectivity in a random graph with density \( \gamma Q \)
evidence for the “edge-attenuation” ansatz

- Results of [Laumann et al., 2010] are easily adapted to show: when $Q = 1$, then almost surely any Hamiltonian model with a connected graph is frustrated if and only if the graph has more than one cycle.
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  \[\ldots\] captures result of [Chvátal+Reed, 1992] on Erdős–Rényi graphs.

- **Claim:** Random Hamiltonians on qubits have a phase transition from unfrustration $\rightarrow$ frustration consistent with this “edge attenuation” ansatz
Erdős–Rényi graphs:

- generalizing [Chvátal+Reed, 1992],
- one may show that “frustrated figure eights” almost surely arise when $\gamma Q > \frac{1}{2}$
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$$x_0 = x_\ell = x_{2\ell}$$

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$$\Rightarrow$$ hard instances of \#2-QSAT on Erdős–Rényi graphs are vanishingly rare, except when $\gamma \in \left(\frac{1}{2}, \frac{1}{2Q}\right)$. 

Percolated lattices:

- pairs of small cycles sharing an edge (“dominoes”) are infinitely abundant in any large component

For $Q > 0$: a constant fraction of these have inconsistent constraints along the three paths $= \Rightarrow$ super-critical random Hamiltonians on percolated lattices are almost surely frustrated.
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\[ \ldots \]
\[ x_{\ell} \cdot x_{\ell+1} \]
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frustration-free models

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  - also: frustration is not relevant to the problem of determining degeneracy of frustration-free systems
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![Diagram of a frustrated system with frozen subsystem]
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![Diagram of a frustrated system]

Niel de Beaudrap (CWI, Amsterdam) Difficult instances of #2-QSAT: very atypical DIQIP/QALGO joint meeting 18 / 23
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A dense enough frustration-free Hamiltonian will almost surely have a large **frozen subsystem**, consisting of qubits whose states are fixed by the Hamiltonian
the frozen subgraph as a “random subgraph”

The **frozen subsystems** form a random subgraph of the random interaction graph.

- (Would-be) frustrations — and thus the frozen graph — emerge at the same time that components with multiple cycles emerge in a random graph with density $\gamma Q$. 

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- **Ansatz.** The frozen graph acts like a random graph with some missing components, and edge-density “similar” to $\gamma Q$. 
emergence and growth of a “frozen core”

- A “giant” component of the frozen subgraph (a frozen core) would decouple the Hamiltonian into independent subsystems.
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- **Open question.** Does the frozen subgraph have a core which grows similarly to the giant component of a random graph with density $\gamma Q$ (where $Q = 1 - \|q\|_2^2$)?

**Proposition.** The frozen subgraph has a core which grows at least as quickly as the giant component of a random graph with edge density $\gamma Q$ (where $Q = 1 - \|q\|_\infty^2$).

**Proof idea.**

- $Q_\infty = \min\left(1 - q_j\right)$ is a lower bound on the probability that an edge incident to a frozen spin will fix the spin on the other end.
- Simulate growth of the frozen subgraph within $G$ by adding edges i.i.d. with probability $Q_\infty$.
- Any frozen subsystem of size $\omega\left(\log n\right)$ can only exist within a “giant” frozen component, whose growth is then bounded from below.
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- **Percolated lattices:** For each \( d \in \{2, 3\} \) there is a percolation constant \( 0 < p_{\text{fin}} < 1 \) such that if \( Q_\infty > p_{\text{fin}} \), the frozen subgraph almost surely decouples Hamiltonians of *any* edge-density into tractible pieces
  
  \( p_{\text{fin}} = \frac{1}{2} \) for square lattices; \( p_{\text{fin}} \geq p_c \approx 0.24881 \) for cubic lattices
Outline

1. Constructing models of random #2-QSAT
   - Erdős–Rényi graphs and percolated lattices
   - Random instances of 2-QSAT on random graphs
   - Random frustration-free Hamiltonians on random graphs
   - Common features of the interaction graph models

2. Analysis of #2-QSAT on random graphs
   - Effective long-range constraints
   - Onset of frustration in random two-body Hamiltonians on qubits
   - Frozen subsystems in frustration-free models

3. Summary
#2-QSAT may be hard in the worst case, but the hard instances are specially constructed.
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