Difficult instances of the counting problem for 2-quantum-SAT are very atypical

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CWI, Amsterdam

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preamble: about frustration-freeness

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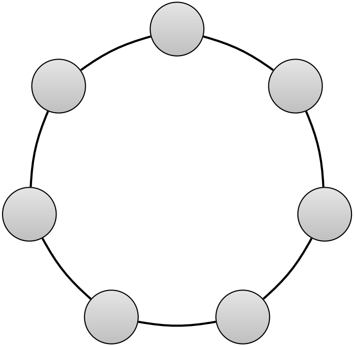

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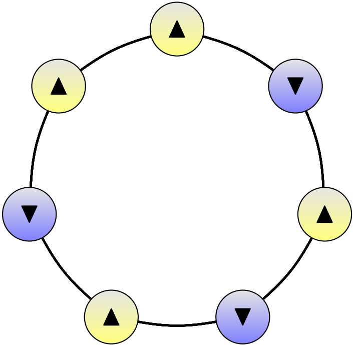

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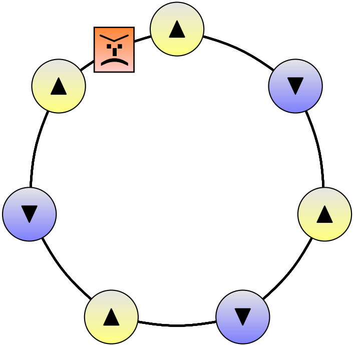

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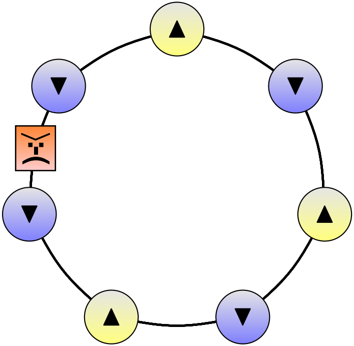

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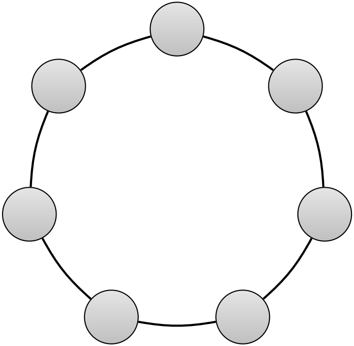

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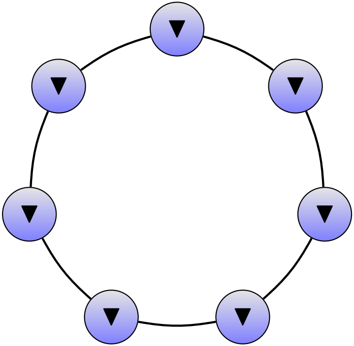

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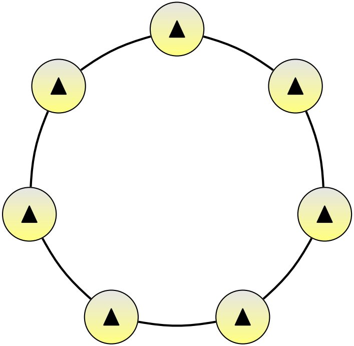

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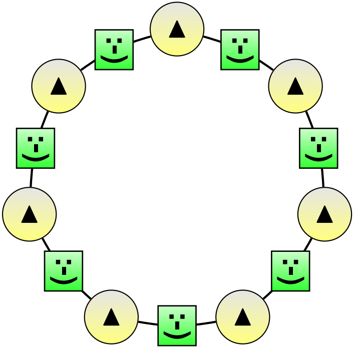

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- (Complexity of **3-SAT**/**3-QSAT** (3-body constraints): . . . very hard.)

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- Generalizes **#2-SAT**: counting solutions to an instance of **2-SAT**

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- More fundamentally: constraints in **multiple local bases**
  - *Monotonicity* seems to be the source of difficulty of **#2-SAT**.
  - The sharp decline in difficulty of **#2-QSAT** provides evidence for this: monotonicity requires a preferred basis.

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Then as $\mathbf{q} \to \mathbf{0}$, 'hard' instances of **#2-QSAT** become unlikely for any fixed density of interaction terms.

Outline

1. Constructing models of random **#2-QSAT**
   - Erdős–Rényi graphs and percolated lattices
   - Random instances of **2-QSAT** on random graphs
   - Random frustration-free Hamiltonians on random graphs
   - Common features of the interaction graph models

2. Analysis of **#2-QSAT** on random graphs
   - Effective long-range constraints
   - Onset of frustration in random two-body Hamiltonians on qubits
   - Frozen subsystems in frustration-free models

3. Summary

models of random graphs (part 1)

Prior work on random 2-satisfiability:

- random **2-SAT** with uniformly random boolean constraints — *e.g.*, **[Chvátal+Reed, 1992]**
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These graphs have "as little structure as possible" —
while perhaps unphysical, this (and prior work) **motivates** this model.

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Results for square/cubic lattices will have analogues for other lattices.

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- **Super-critical graph:** largest component is "complicated" and has size $O(n)$ for $\gamma > \gamma_0$: *interesting* instances.

Outline

1. Constructing models of random **#2-QSAT**
   - Erdős–Rényi graphs and percolated lattices
   - Random instances of **2-QSAT** on random graphs
   - Random frustration-free Hamiltonians on random graphs
   - Common features of the interaction graph models

2. Analysis of **#2-QSAT** on random graphs
   - Effective long-range constraints
   - Onset of frustration in random two-body Hamiltonians on qubits
   - Frozen subsystems in frustration-free models

3. Summary

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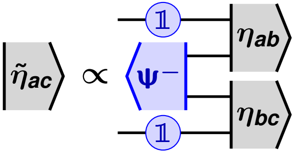

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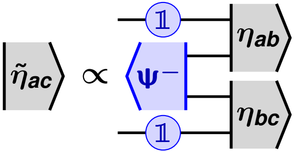

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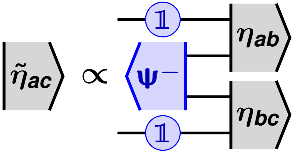

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Ansatz: long range constraints in a Hamiltonian with density $\gamma > 0$
act like connectivity in a random graph with density γQ

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- Results of [Laumann *et al.*, 2010] are easily adapted to show: when $Q = 1$, then almost surely any Hamiltonian model with a connected graph is frustrated if and only if the graph has more than one cycle.

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- **Claim:** Random Hamiltonians on qubits have a phase transition from unfrustration $\to$ frustration consistent with this “edge attenuation” ansatz

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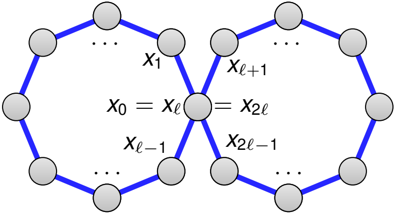

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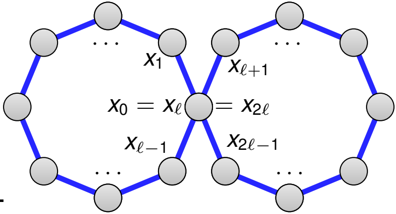

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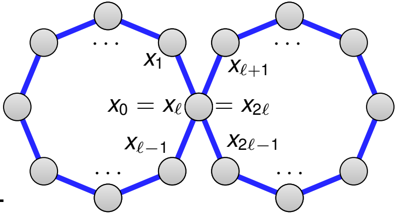

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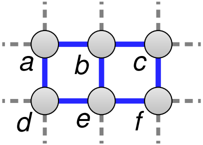

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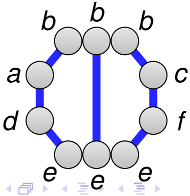

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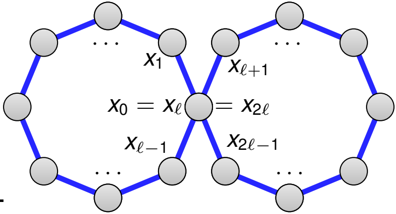

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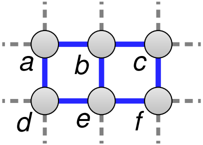

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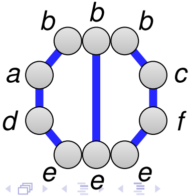

$\Longrightarrow$ super-critical random Hamiltonians on percolated lattices are almost surely frustrated.

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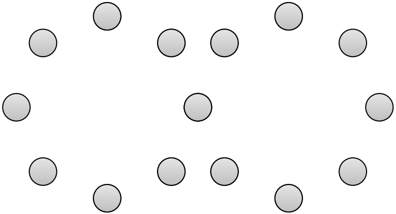

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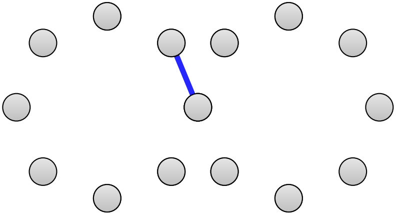

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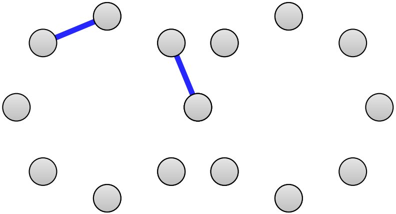

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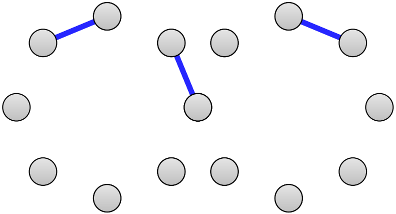

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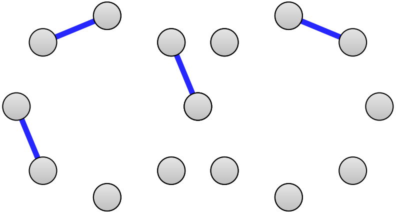

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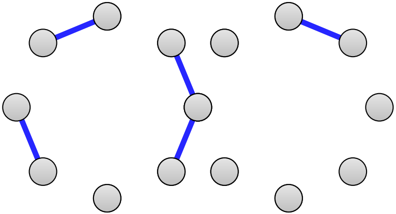

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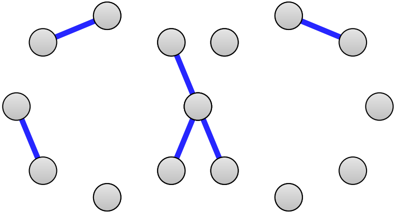

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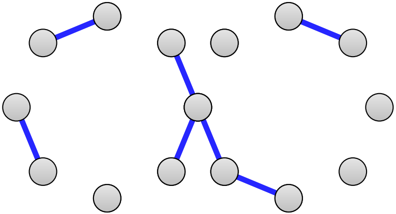

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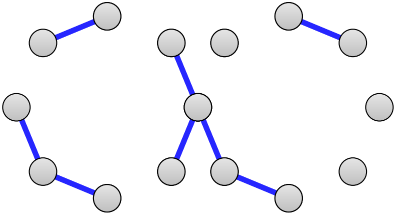

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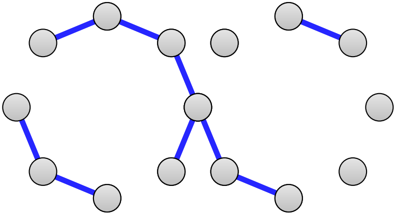

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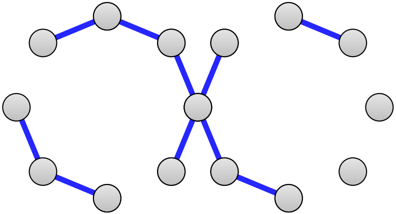

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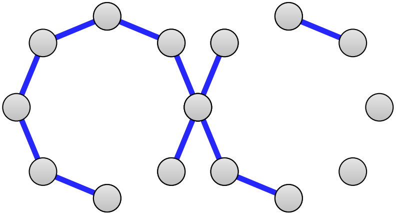

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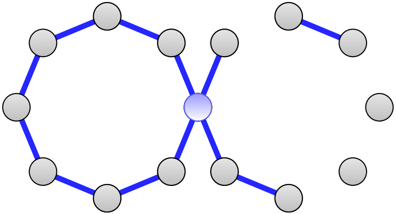

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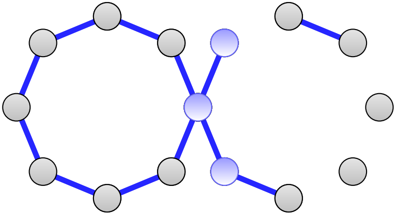

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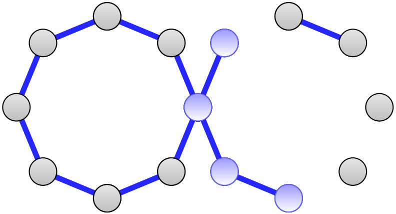

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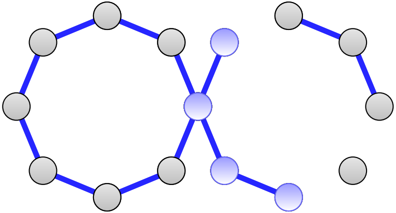

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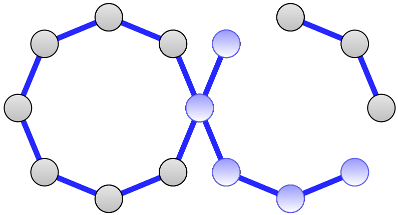

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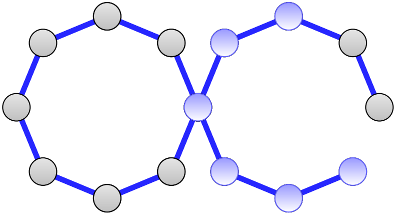

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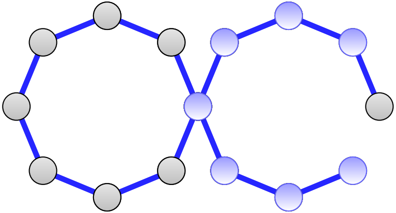

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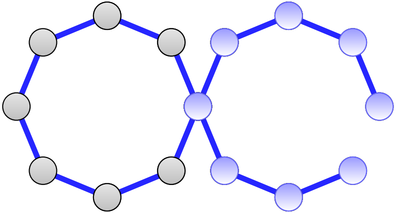

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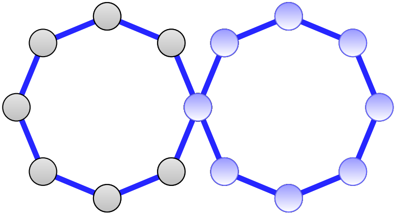

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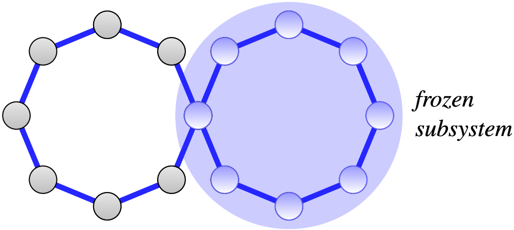

- A dense enough frustration-free Hamiltonian will almost surely have a large **frozen subsystem**, consisting of qubits whose states are fixed by the Hamiltonian

the frozen subgraph as a “random subgraph”

The **frozen subsystems** form a random subgraph of the random interaction graph.

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- **Ansatz.** The frozen graph acts like a random graph with some missing components, and edge-density “similar” to γQ.

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  - Simulate growth of the frozen subgraph within G by adding edges *i.i.d.* with probability Q_∞.
  - Any frozen subsystem of size $\omega(\log n)$ can only exist within a “giant” frozen component, whose growth is then bounded from below.

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- **Percolated lattices:** For each $d \in \{2, 3\}$ there is a percolation constant $0 < p_{\mathrm{fin}} < 1$ such that if $Q_\infty > p_{\mathrm{fin}}$, the frozen subgraph almost surely decouples Hamiltonians of *any* edge-density into tractible pieces

  ($p_{\mathrm{fin}} = \frac{1}{2}$ for square lattices; $p_{\mathrm{fin}} \geqslant p_c \approx 0.24881$ for cubic lattices)

Outline

1. Constructing models of random **#2-QSAT**
   - Erdős–Rényi graphs and percolated lattices
   - Random instances of **2-QSAT** on random graphs
   - Random frustration-free Hamiltonians on random graphs
   - Common features of the interaction graph models

2. Analysis of **#2-QSAT** on random graphs
   - Effective long-range constraints
   - Onset of frustration in random two-body Hamiltonians on qubits
   - Frozen subsystems in frustration-free models

3. Summary

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